Last class:

Mean Value Theorem for Integrals

$f: [a,b] \to \mathbb{R}$ cont.

$\Rightarrow \exists\, y \in (a,b)$ s.t. $f(y) = \frac{1}{b-a} \int_a^b f(x)\,dx$

"average value of f on $[a,b]$"

Theorem Let $a < c < b$

Assume f is integrable on $[a,c]$ and $[c,b]$

$\Rightarrow$ f integrable on $[a,b]$ and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Proof. f integrable on $[a,c]$

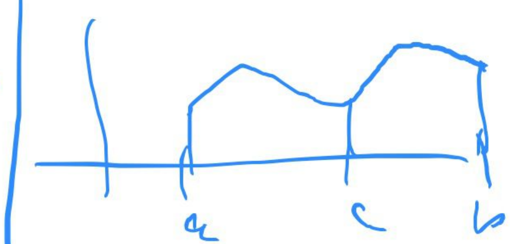
$\Rightarrow \exists$ partition P_1 of $[a,c]$ s.t. $U(f,P_1) - L(f,P_1) < \varepsilon/2$

f integrable on $[c,b]$

$\Rightarrow \exists$ partition P_2 of $[c,b]$ s.t. $U(f,P_2) - L(f,P_2) < \varepsilon/2$

$\Rightarrow$ $P = P_1 \cup P_2$ is a partition of $[a,b]$

such that



$$\mathcal{U}(f,P) - L(f,P) = \mathcal{U}(f,P_1) + \mathcal{U}(f,P_2) - L(f,P_1) - L(f,P_2)$$

$$= \underbrace{\mathcal{U}(f,P_1) - L(f,P_1)}_{<\varepsilon/2} + \underbrace{\mathcal{U}(f,P_2) - L(f,P_2)}_{<\varepsilon/2}$$

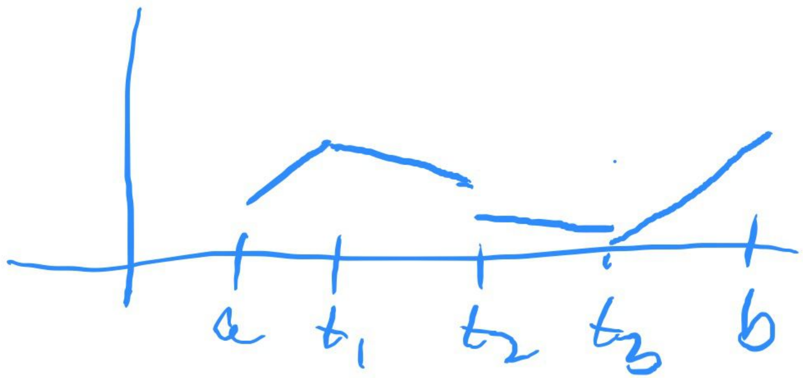
$$< \varepsilon$$

$\Rightarrow$ claim.

have already seen: any monotonic functions is integrable.
Can be generalized

Def. A function $f: [a,b] \to \mathbb{R}$ is called piecewise monotonic if we can find a partition $P = \{t_0, t_1, \dots t_n\}$ of $[a,b]$ such that $f|_{(t_{k-1}, t_k)}$ is monotonic

Example



idea: extend f from (t_{k-1}, t_k) to $\tilde{f}$ defined on $[t_{k-1}, t_k]$ which preserves monotonicity.

example: if $f|_{(t_{k-1}, t_k)}$ mon. decreasing

define $\tilde{f}(t_{k-1}) = \sup\{f(x),\ x \in (t_{k-1}, t_k)\}$

$\tilde{f}(t_k) = \inf\{f(x),\ x \in [t_{k-1}, t_k]\}$

Remark: By homework problem 32.7 the integral $\int_a^b f(x)\,dx$ does not change if we vary the function f at finitely many points

$\Rightarrow$ even if $f(t_{k-1}) \neq \tilde{f}(t_{k-1})$ or $f(t_k) \neq \tilde{f}(t_k)$

we have $\int_{t_{k-1}}^{t_k} f(x)\,dx = \int_{t_{k-1}}^{t_k} \tilde{f}(x)\,dx$ (if integral exists!)

As $\tilde{f}|_{[t_{k-1}, t_k]}$ is monotonic $\Rightarrow$ $\int_{t_{k-1}}^{t_k} \tilde{f}(x)\,dx$ exists

$\Rightarrow$ $\int_{t_{k-1}}^{t_k} f(x)\,dx$ exists

By previous theorem

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(x)\,dx$$

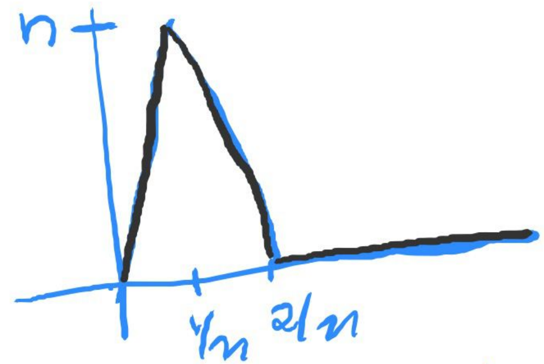
✓

Question: Assume $f_n \to f$ pointwise on $[a,b]$

can we conclude $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$?

Answer in general: NO

Example: let f_n be given by following graph



$$\Rightarrow \int_0^1 f_n(x)\,dx = \text{area of triangle} = \underset{\substack{\uparrow\\ \text{height}}}{\frac{1}{2}\, n} \cdot \underset{\substack{\uparrow\\ \text{width}}}{\left(\frac{2}{n}\right)} = 1$$

BUT: $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$

indeed: obviously true if $x=0$

if $x>0$

$\Rightarrow$ $f_n(x)=0$ whenever $x \geq \frac{2}{n} \Leftrightarrow n \geq \frac{2}{x}$

$\Rightarrow$ $\lim_{n\to\infty} f_n(x)=0$

$\Rightarrow$ $f_n \longrightarrow f =$ zero function pointwise

but $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq 0 = \int_0^1 f(x)\,dx$

Positive Results

Dominated Convergence Theorem

Assume $f_n \to f$ pointwise on $[a,b]$ and there exists a uniform bound M (i.e. $|f_n(x)| \leq M$ $\forall n$, $\forall x \in [a,b]$)

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Monotone Convergence Theorem

Assume $f_n \to f$ pointwise on $[a,b]$, all f_n's monotonic

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Fundamental Theorem(s) of Calculus

we consider 2 versions:

informally: I g differentiable

$\Rightarrow$ integral of g' $\rightarrow g$

II f continuous

$\Rightarrow$ derivative of $F(x) = \int_0^x f(t)\,dt = f$

in both cases differentiation and integration cancel each other.

Fundamental Theorem of Calculus I

Assume $g : [a,b] \to \mathbb{R}$ satisfies

- differentiable in (a,b)
- $g'(x)$ integrable

$$\Rightarrow \int_a^b g'(x)\,dx = g(b) - g(a)$$

Proof. Let $\varepsilon > 0$

g' is integrable $\Rightarrow \exists$ partition $P = \{t_0, t_1, \dots, t_n\}$

such that $U(g', P) - L(g', P) < \varepsilon$

apply MVT for interval $[t_{k-1}, t_k]$:

$$\Rightarrow \exists\, x_k \in (t_{k-1}, t_k) \text{ s.t. } \quad g'(x_k) = \frac{1}{(t_k - t_{k-1})} \int_{t_{k-1}}^{t_k} g'(x)\,dx \Leftarrow$$

$\Leftrightarrow \quad g'(x_k)(t_k - t_{k-1}) = g(t_k) - g(t_{k-1})$

$\Rightarrow \quad g(b) - g(a) = \sum_{k=1}^{n} g(t_k) - g(t_{k-1}) \qquad \left(\begin{array}{l} t_n = b \\ t_0 = a \end{array} \right)$

$$\boxed{g(b) - g(a) = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1})}$$

$L(g', P) = \sum_{k=1}^{n} \underbrace{m(g', [t_{k-1}, t_k])}_{\inf \{ g'(x), x \in \}} (t_k - t_{k-1})$

$U(g', P) = \sum_{k=1}^{n} M(\ldots)(t_k - t_{k-1})$

we have $m(g', [\ ]) \leq g'(x_k) \leq M(g', [\ ])$ $\qquad$ $| \cdot (t_k - t_{k-1})$ and sum over k.

$$\Longrightarrow \quad \boxed{L(g', P) \leq g(b) - g(a) \leq U(g', P)}$$

by def we also have

$$\boxed{L(g', P) \le \int_a^b g'(x)\,dx \le U(g', P)}$$